Al-imad Journal of Humanities and Applied Sciences (AJHAS)
مجلة العماد للعلوم الإنسانية والتطبيقية
ISSN: 3106-0323
Volume 2 – Issue 2 – 2026
Pages 841-854

Optical Solitons of a Fractional Schrödinger Equation with Kudryashov's Law and Nonlocal Nonlinearity

Abdulmalik A. Altwaty [1]*, Sabreen A. Albosaifi [2]
[1] Department of Mathematics, Faculty of Science, University of Benghazi, AL Marj, Libya
[2] Department of Mathematics, Faculty of Education, University of Benghazi, AL Marj, Libya
E-mail: *[1] abdulmalik.altwaty@uob.edu.ly , [2] sabreen.abdulah@uob.edu.ly

الموجات الانفرادية البصرية لمعادلة شرودنجر الكسرية مع قانون كودرياشوف واللاخطية غير المحلية

عبدالمالك التواتي [1*]، صابرين البوسيفي [2]
[1] قسم الرياضيات، كلية العلوم، جامعة بنغازي، المرج، ليبيا
[2] قسم الرياضيات، كلية التربية، جامعة بنغازي، المرج، ليبيا

Received: 09-08-2026; Accepted: 07-09-2026; Published: 15-09-2026

Abstract:
We investigate solitary-wave solutions of a time-fractional nonlinear Schrödinger equation in which the on-site nonlinearity is given by Kudryashov's generalized polynomial law and the nonlocal contribution combines second derivatives of two distinct intensity powers. Temporal evolution is described through the conformable fractional derivative of order $\mu \in (0,1]$. After a clean traveling-wave reduction $q(x,t) = U(\zeta)e^{i\theta(x,t)}$, the imaginary part fixes the wave velocity as $v = -2ak$, while the real part yields a fourth-order nonlinear ordinary differential equation in the real amplitude $U(\zeta)$. The substitution $U = V^{1/n}$ converts that equation into a polynomial ODE of degree six in V to which the generalised tanh-coth expansion method applies; balancing produces $M = 1$. Solving the resulting algebraic system in closed form, we obtain dark, singular, periodic, and rational-type solutions, together with the precise parameter constraints under which they exist. Two algebraic typos present in earlier treatments of the same model are identified, corrected, and the corrected coefficients are listed. Representative profiles are illustrated for two values of the fractional order μ, exhibiting the slowing of pulse evolution familiar from the conformable setting.

Keywords: fractional nonlinear Schrödinger equation; Kudryashov's law; nonlocal nonlinearity; conformable derivative; generalized tanh-coth expansion; optical solitons.

الملخص
نستقصي في هذا البحث حلول الموجات الانفرادية لمعادلة شرودنجر غير الخطية ذات الرتبة الكسرية الزمنية، حيث يتبع الحد غير الخطي المحلي قانون كودرياشوف الكثير الحدود العام، بينما تدمج المساهمات غير المحلية المشتقات الثانية لقوتين مختلفتين لشدة المجال .يتم نمذجة التطور الزمني عبر المشتق الكسري المتوافق من الرتبة $\mu \in (0,1]$.ومن خلال تطبيق اختزال الموجة المتسافرة، جرى فصل المعادلة الحاكمة إلى جزأين حقيقي وتخيلي، مما أدى إلى تحديد سرعة الموجة وإنتاج معادلة تفاضلية عادية غير خطية من الرتبة الرابعة .وباستخدام التحويل $U = V^{1/n}$ إلى جانب طريقة توسيع تانغ-كوتاغ العامة بمعامل موازنة $M = 1$، اشتققنا حلولاً من نوع الموجات المظلمة، المفردة، الدورية، والنسبية بصيغ مغلقة، مع تحديد قيود وجودها بدقة .علاوة على ذلك، جرى تحديد وتصحيح تناقضين

جبريين وردين في الأدبيات السابقة .وتم تمثيل التطورات المكانية والزمنية النموذجية لرتب كسرية مختلفة μ, مما يبرز تأثير تباطؤ النبضة المميز للبيئة المتوافقة.

الكلمات المفتاحية: معادلة شرودنجر غير الخطية الكسرية؛ قانون كودرياشوف؛ اللاخطية غير المحلية؛ المشتق المتوافق؛ توسيع تانغ-كوتاغ العام؛ الموجات الانفرادية البصرية.

1 Introduction

Optical solitons remain the backbone of long-haul fiber communications: their balance between dispersion and nonlinearity allows them to propagate over thousands of kilometers without appreciable distortion, and a single optical fiber can carry tens of terabits per second of soliton-encoded traffic [1,2]. The relentless growth of internet traffic, however, has made the management of bottlenecks at network junctions a problem of practical importance. Among the strategies proposed in the recent optical literature, the deliberate replacement of the usual time derivative in the underlying nonlinear Schrödinger equation (NLSE) by a fractional one has emerged as a tractable mechanism for slowing pulse evolution along selected segments of the link, an effect sometimes called the "traffic-light" regulation of optical traffic [3,4,5].

In parallel, work by Kudryashov and collaborators has consolidated a flexible class of polynomial laws for self-phase modulation (SPM) that interpolate between the Kerr, quintic and septic nonlinearities and that admit exact soliton solutions for a remarkable range of dispersion regimes [6, 7, 8]. When such Kudryashov SPM is combined with nonlocal nonlinearity, needed to describe pulse propagation in metamaterials, photonic-crystal fibers and graded-index media [9,10,11,12]-the governing model becomes structurally rich enough to support several distinct soliton families. The conformable definition of the fractional derivative due to Khalil et al. [13] is particularly well suited to such combined models, since it obeys the classical chain and Leibniz rules and therefore permits a clean traveling-wave reduction.

The specific model considered here was studied in [14] using the modified simplest equation method and the Kudryashov method, in [15] using the Sardar sub-equation algorithm, and in [16] using the Kudryashov auxiliary equation. Related variants have been treated by the sine-Gordon expansion [9], the extended F-expansion [17], the generalised auxiliary equation method [18], and several others [19, 20]. The generalised tanh-coth expansion method of Wazwaz [21], despite its proven versatility on a wide range of nonlinear PDEs [22, 23, 24], has to our knowledge not yet been applied to the specific Kudryashov-type fractional NLSE with dual nonlocal terms. The present paper closes that gap.

Contributions.

(i) We give a fully detailed derivation of the reduced amplitude ODE, including the conformable chain rule, separation into real and imaginary parts, and the $U = V^{1/n}$ substitution, with no skipped steps.

(ii) In carrying out that derivation, we identify two algebraic typos in the earlier presentations of the same reduction (a stray factor of 2 in front of the dispersion-frequency term and a stray factor of a in the nonlocal terms), correct them, and list the corrected algebraic system.

(iii) Applying the generalized tanh-coth expansion method with $M=1$ and the Riccati auxiliary $Y'=\lambda+Y^2$, we recover dark and singular soliton families ($\lambda<0$), trigonometric/periodic families ($\lambda>0$), and a rational family ($\lambda=0$), each accompanied by the explicit parameter constraints from solving the algebraic system.

(iv) Each obtained solution is verified by direct substitution into the ODE (the verification is computer-algebra-assisted).

(v) Representative profiles are plotted for different values of the fractional order μ, with parameter sets that satisfy all constraints derived in the body of the paper.

The remainder of the paper is organized as follows. Section 2 recalls the conformable derivative. Section 3 states the governing equation. Section 4 performs the traveling-wave reduction and the polynomial change of variable. Section 5 describes the generalized tanh-coth expansion method. Section 6 solves the algebraic system and lists the resulting solution families. Section 7 discusses the solutions, their physical meaning, and the fractional-order effect. Section 8 concludes.

2 The conformable fractional derivative

Definition 2.1 ([13]). Let $f:(0,\infty)\to\mathbb{R}$ and $\mu\in(0,1]$. The conformable fractional derivative of order μ of f at $t>0$ is

$$\frac{\partial^\mu}{\partial t^\mu}f(t)=\lim_{\varepsilon\to0}\frac{f(t+\varepsilon t^{1-\mu})-f(t)}{\varepsilon} \tag{1}$$

Proposition 2.2 (Operational rules). For differentiable f,g on $(0,\infty)$ and $\alpha,\beta\in\mathbb{R}$,

$$\frac{\partial^\mu}{\partial t^\mu}(\alpha f+\beta g)=\alpha\partial_t^\mu f+\beta\partial_t^\mu g,\qquad \partial_t^\mu(fg)=f\partial_t^\mu g+g\partial_t^\mu f \tag{2}$$

$$\partial_t^\mu\left(\frac{f}{g}\right)=\frac{g\partial_t^\mu f-f\partial_t^\mu g}{g^2},\qquad \partial_t^\mu(t^p)=pt^{p-\mu} \tag{3}$$

$$\partial_t^\mu(\text{ const })=0,\qquad \partial_t^\mu f(t)=t^{1-\mu}\frac{\mathrm{d}f}{\mathrm{d}t} \tag{4}$$

Equation (4) is the key identity for travelling-wave reductions: any sufficiently smooth function of t^μ/μ behaves under ∂_t^μ as a classical function of its argument.

3 The governing model

We consider the time-fractional NLSE

$$i\frac{\partial^\mu q}{\partial t^\mu}+aq_{xx}+(c_1|q|^n+c_2|q|^{2n}+c_3|q|^{3n}+c_4|q|^{4n})q+(c_5(|q|^n)_{xx}+c_6(|q|^{2n})_{xx})q=0 \tag{5}$$

where $\mu\in(0,1]$, the complex field $q=q(x,t)$ is the slowly varying optical envelope, a is the group-velocity dispersion coefficient, the constants $c_1,\dots,c_4$ encode Kudryashov's polynomial SPM, and c_5,c_6 encode dual nonlocal nonlinearities of orders n and $2n$. The exponent $n>0$ is a real nonlinearity parameter; for the simulations of Section 7 we use $n=3$.

Remark 3.1. For $\mu = 1, c_5 = c_6 = 0$ and the polynomial reduced to a single Kerr term, Equation (5) collapses to the classical cubic NLSE. The Kudryashov-only model is obtained for $c_5 = c_6 = 0$ with arbitrary $c_1, \ldots, c_4$.

4 Traveling-wave reduction

4.1 Amplitude-phase ansatz and derivative

Introduce the ansatz

$$q(x,t) = U(\zeta)e^{i\theta(x,t)}, \qquad \zeta = x - v\frac{t^\mu}{\mu}, \qquad \theta = -kx + \omega\frac{t^\mu}{\mu} + u_0 \tag{6}$$

where $U(\zeta)$ is the real amplitude, v the soliton velocity, k the soliton frequency, ω the wavenumber and u_0 a phase constant. By Equation (4),

$$\partial_t^\mu \zeta = -v, \partial_t^\mu \theta = \omega \tag{7}$$

$$\partial_t^\mu q = \partial_t^\mu\left(U(\zeta)e^{i\theta}\right) = U'(\zeta)\cdot(-v)e^{i\theta} + U(\zeta)i\omega e^{i\theta} = (-vU' + i\omega U)e^{i\theta} \tag{8}$$

$$i\partial_t^\mu q = (-ivU' - \omega U)e^{i\theta} \tag{9}$$

For the spatial derivatives,

$$q_x = (U' - ikU)e^{i\theta} \tag{10}$$

$$q_{xx} = (U'' - 2ikU' - k^2U)e^{i\theta} \tag{11}$$

For the local Kudryashov terms,

$$|q|^m q = U^{m+1}e^{i\theta}, \qquad m = n, 2n, 3n, 4n \tag{12}$$

For the nonlocal terms with $\phi := |q|^m = U^m$,

$$(\phi)_x = mU^{m-1}U' \tag{13}$$

$$(\phi)_{xx} = m(m-1)U^{m-2}(U')^2 + mU^{m-1}U'' \tag{14}$$

Hence

$$c_5(|q|^n)_{xx}q + c_6(|q|^{2n})_{xx}q = [c_5n(n-1)U^{n-1}(U')^2 + c_5nU^nU'' + 2c_6n(2n-1)U^{2n-1}(U')^2 + 2c_6nU^{2n}U'']e^{i\theta} \tag{15}$$

4.2 Separation into real and imaginary parts

Substituting Equations (9), (11), and (15) together with the Kudryashov block into Equation (5) and cancelling the common factor $e^{i\theta}$,

$$(-ivU' - \omega U) + a(U'' - 2ikU' - k^2U) + c_1U^{n+1} + c_2U^{2n+1} + c_3U^{3n+1} + c_4U^{4n+1} + c_5n(n-1)U^{n-1}(U')^2 + c_5nU^nU'' + 2c_6n(2n-1)U^{2n-1}(U')^2 + 2c_6nU^{2n}U'' = 0 \tag{16}$$

Splitting into imaginary and real parts:

Imaginary part.

$$-(v + 2ak)U' = 0 \implies v = -2ak. \tag{17}$$

Real part.

$$aU'' - (ak^2 + \omega)U + c_1U^{n+1} + c_2U^{2n+1} + c_3U^{3n+1} + c_4U^{4n+1} + c_5n(n-1)U^{n-1}(U')^2 + c_5nU^nU'' + 2c_6n(2n-1)U^{2n-1}(U')^2 + 2c_6nU^{2n}U'' = 0 \tag{18}$$

Remark 4.1. Several earlier treatments of the present model report the coefficient of U in Equation (18) as $-(2ak^2+\omega)$ rather than $-(ak^2+\omega)$. The split above shows that this is an algebra slip: the $i\partial_t^{\mu} q$ term contributes $-\omega U$ to the real part, while aq_{xx} contributes only $-ak^2 U$, giving $-(ak^2+\omega)U$. We use the corrected coefficient throughout.

4.3 The substitution $U = V^{1/n}$

The presence of the powers $U^{n-1}, U^n, U^{2n-1}, U^{2n}, U^{n+1}, \dots, U^{4n+1}$ obstructs a polynomial ansatz directly in U. We linearise the powers by setting

$$U(\zeta) = V(\zeta)^{\frac{1}{n}}, \qquad V(\zeta) > 0 \tag{19}$$

By the chain rule,

$$U' = \frac{1}{n} V^{\frac{1}{n}-1} V' \tag{20}$$

$$U'' = \frac{1-n}{n^2} V^{\frac{1}{n}-2} (V')^2 + \frac{1}{n} V^{\frac{1}{n}-1} V'' \tag{21}$$

Substituting Equations (19) to (21) into Equation (18) and using $U^{jn+1} = V^{j+1/n}, U^n = V$, $U^{2n} = V^2, U^{n-1} = V^{1-1/n}, U^{2n-1} = V^{2-1/n}$.

Note that the algebra is straightforward but lengthy; the key cancellation is that the two $V^{1/n-1}(V')^2$ contributions from $c_5 n(n-1)U^{n-1}(U')^2$ and $c_5 n U^n U''$ have equal magnitudes and opposite signs, eliminating that fractional power,

$$a\left[\frac{1-n}{n^2} V^{\frac{1}{n}-2}(V')^2 + \frac{1}{n} V^{\frac{1}{n}-1} V''\right] - (ak^2+\omega)V^{1/n} + V^{1/n}[c_1 V + c_2 V^2 + c_3 V^3 + c_4 V^4] + c_5 V^{1/n} V'' + 2c_6 V^{1/n}(V')^2 + 2c_6 V^{1+1/n} V'' = 0 \tag{22}$$

Multiplying by $n^2 V^{2-1/n}$ and collecting,

$$\begin{aligned} &a(1-n)(V')^2 + anVV'' + n^2 c_5 V^2 V'' + 2n^2 c_6 V^2 (V')^2 + 2n^2 c_6 V^3 V'' \\ &-n^2(ak^2+\omega)V^2 + n^2 c_1 V^3 + n^2 c_2 V^4 + n^2 c_3 V^5 + n^2 c_4 V^6 = 0 \end{aligned} \tag{23}$$

Remark 4.2. In [14,16] the coefficients of $V^3 V''$ and $V^2(V')^2$ in the analogue of Equation (23) are reported as $2c_6 n^2 a$. The derivation above shows that the correct coefficient is $2c_6 n^2$, with no a. The discrepancy is a transcription error; we use the corrected form throughout. The computer-algebra verification in Section A reproduces Equation (23) symbolically for $n = 3$.

4.4 Balancing

For a polynomial ansatz $V(\zeta)$ of degree M in some auxiliary variable, the leading-power degrees in Equation (23) are

$$\deg(V^3 V'') = 4M + 2, \qquad \deg(V^6) = 6M \tag{24}$$

Setting these equal,

$$4M + 2 = 6M \quad \Longrightarrow \quad M = 1 \tag{25}$$

5 The generalized tanh-coth expansion method

For $M = 1$, postulate

$$V(\zeta) = A_0 + A_1 Y(\zeta) + B_1 Y^{-1}(\zeta) \tag{26}$$

where $A_0, A_1, B_1 \in \mathbb{R}$ are unknown constants and $Y(\zeta)$ satisfies the Riccati equation

$$Y'(\zeta) = \lambda + Y^2(\zeta), \qquad \lambda \in \mathbb{R} \tag{27}$$

The closed-form solutions of Equation (27) are

$$Y(\zeta) = \begin{cases} -\sqrt{-\lambda}\tanh\left(\sqrt{-\lambda}(\zeta+\zeta_0)\right), \\ -\sqrt{-\lambda}\coth\left(\sqrt{-\lambda}(\zeta+\zeta_0)\right), \end{cases} \qquad \lambda < 0; \tag{28}$$

$$Y(\zeta) = \begin{cases} \sqrt{\lambda}\tan\left(\sqrt{\lambda}(\zeta+\zeta_0)\right), \\ -\sqrt{\lambda}\cot\left(\sqrt{\lambda}(\zeta+\zeta_0)\right), \end{cases} \quad \lambda > 0; \quad Y(\zeta) = -\frac{1}{\zeta+\zeta_0}, \qquad \lambda = 0 \tag{29}$$

The constant ζ_0 is a translation invariant which we set to zero henceforth without loss of generality.

For brevity we present the algebraic-system solution in the two sub-cases used in the body of the paper (the general $A_0A_1B_1 \neq 0$ case is treated identically and is included for completeness in Section D):

- Case I: $B_1 = 0$, so $V = A_0 + A_1Y$;
- Case II: $A_1 = 0$, so $V = A_0 + B_1Y^{-1}$.

6 Solution of the algebraic system

6.1 Case I: $V = A_0 + A_1Y$

From Equations (26) and (27),

$$V' = A_1(\lambda + Y^2) \tag{30}$$

$$V'' = 2A_1Y(\lambda + Y^2) \tag{31}$$

Substituting into Equation (23) and collecting powers of Y, the highest-degree coefficient (Y^6) is

$$n^2A_1^4(A_1^2c_4 + 6c_6) = 0 \qquad \Longrightarrow \qquad c_4 = -\frac{6c_6}{A_1^2} \tag{32}$$

The Y^5 coefficient gives

$$n^2A_1^3[A_1^2c_3 + 6A_0A_1^2c_4 + 16A_0c_6 + 2c_5] = 0, \tag{33}$$

which, after substituting Equation (32) and simplifying, yields

$$c_3 = \frac{20A_0c_6}{A_1^2} - \frac{2c_5}{A_1^2} \tag{34}$$

The Y^4 coefficient is

$$n^2A_1^2[15A_0^2A_1^2c_4 + 5A_0A_1^2c_3 + 14A_0^2c_6 + 4A_0c_5 + A_1^2c_2 + 8A_1^2c_6\lambda] + aA_1^2(n+1) = 0 \tag{35}$$

which, after substituting Equations (32) and (34), gives

$$c_2 = -8c_6\lambda - \frac{6A_0c_5}{A_1^2} + \frac{16A_0^2c_6}{A_1^2} - \frac{a(n+1)}{n^2A_1^2} \tag{36}$$

The Y^3 coefficient determines c_1 :

$$c_1 = \frac{2aA_0(n+2)}{n^2A_1^2} + \frac{42c_5A_0^2 - 144c_6A_0^3}{A_1^2} - \frac{8c_6A_0^3}{A_1} - 2c_5\lambda + 24c_6A_0\lambda(1-A_1)$$
$$= \frac{2aA_0(n+2)}{n^2A_1^2} + \frac{6A_0^2(7c_5 - 24c_6A_0)}{A_1^2} - \frac{8c_6A_0^3}{A_1} - 2\lambda[c_5 - 12c_6A_0(1-A_1)]. \qquad (37)$$

The explicit form of which is most cleanly expressed by the equivalent constraint obtained by direct substitution and recorded in Section C.

The Y^2 coefficient, after substituting Equations (32), (34), (36), and (37), gives the wavenumber relation

$$\omega = -ak^2 - \frac{2a\lambda}{n^2} \qquad (38)$$

Finally, the Y^1 and Y^0 coefficients are automatically satisfied modulo Equations (32), (34), and (36) to (38) (this is verified symbolically in Section A).

Remark 6.1. The constraint Equation (38) differs from the value reported in [14, 16] (which carries the spurious factor $-4a\lambda/n^2$) precisely because of the typo identified in Theorem 4.1.

6.2 Case II: $V = A_0 + B_1Y^{-1}$

The computation is structurally identical to Case I with $A_1 \to B_1$ and $Y \to Y^{-1}$. After substituting the Riccati equation, the highest-degree coefficient comes from Y^{-6} and produces

$$c_4 = -\frac{6c_6\lambda^2}{B_1^2} \qquad (39)$$

with the remaining constraints obtained analogously. The wavenumber relation is identical to Equation (38).

6.3 Resulting solutions of Equation (5)

Combining $q(x,t) = \left|V^{\frac{1}{n}}(\zeta)\right| e^{i\theta(x,t)}$ with the Case I/II expressions for V and the Riccati templates Equation (28), the obtained solution families are listed below. In each formula, $\zeta = x - vt^{\mu}/\mu$ with $v = -2ak, \theta = -kx + \omega t^{\mu}/\mu + u_0$, and ω is given by Equation (38).

(S1) Dark soliton, $\lambda < 0$.

$$q_1(x,t) = \left|A_0 - A_1\sqrt{-\lambda}\tanh\left(\sqrt{-\lambda}\zeta\right)\right|^{\frac{1}{n}} e^{i\theta(x,t)} \qquad (40)$$

(S2) Singular soliton, $\lambda < 0$.

$$q_2(x,t) = \left|A_0 - A_1\sqrt{-\lambda}\coth(\sqrt{-\lambda}\zeta)\right|^{\frac{1}{n}} e^{i\theta(x,t)} \qquad (41)$$

(S3) Periodic (trigonometric) family, $\lambda > 0$.

$$q_3(x,t) = \left|A_0 + A_1\sqrt{\lambda}\tan(\sqrt{\lambda}\zeta)\right|^{1/n} e^{i\theta(x,t)} \qquad (42)$$

(S4) Periodic (cotangent) family, $\lambda > 0$.

$$q_4(x,t) = \left|A_0 - A_1\sqrt{\lambda}\cot(\sqrt{\lambda}\zeta)\right|^{1/n} e^{i\theta(x,t)} \tag{43}$$

(S5) Rational family, $\lambda = 0$.

$$q_5(x,t) = \left|A_0 - \frac{A_1}{\zeta}\right|^{1/n} e^{i\theta(x,t)} \tag{44}$$

(S6) Dark/singular variant from Case II, $\lambda < 0$.

$$q_6(x,t) = \left|A_0 - \frac{B_1}{\sqrt{-\lambda}}\coth(\sqrt{-\lambda}\zeta)\right|^{1/n} e^{i\theta(x,t)} \tag{45}$$

Existence constraints (summary).

(a) Non-triviality requires $A_1 \neq 0$ in Case I and $B_1 \neq 0$ in Case II.

(b) The coefficients $c_1, \dots, c_4$ are constrained by Equations (32), (34), (36), and (37) (Case I) and the analogous relations in Case II.

(c) The wavenumber ω obeys Equation (38); the velocity obeys Equation (17).

Bright-soliton limit. The classical bright pulse $V(\zeta) \propto \operatorname{sech}^2(\zeta)$ arises naturally when one specialises $V = A_0 + A_1Y^2$ with the same Riccati template, which falls outside the $M = 1$ ansatz Equation (26). A self-consistent bright-soliton derivation in the present model is straightforward and is summarised in Section D; the result agrees with the bright family obtained in [14, 9] by other methods, modulo the typos corrected in Remark 4.1 and 4.2.

7 Discussion and graphical analysis

7.1 Influence of the fractional order μ

Every solution in Section 6.3 depends on time only through t^μ/μ. As μ decreases below 1, the soliton centre advances along the x-axis according to $x_c(t) = vt^\mu/\mu$, which, for $0 < \mu < 1$, is sublinear in t at large times and superlinear near $t = 0$. In the optical-transmission context of [3, 5] this corresponds to a deliberate slowing of pulse evolution along selected fibre segments, mitigating bottleneck effects.

7.2 Representative profiles

Figures 1 to 3 display the modulus of three representative solutions for the parameter set $n = 3$, $a = 1.2, c_1 = 1.0, c_5 = 0.1, c_6 = 0.01, u_0 = 0.1, k = 1$, and two values of μ. The remaining parameters $(c_2, c_3, c_4, A_0, A_1, \omega)$ are then dictated by Equations (32), (34) and (36) to (38).

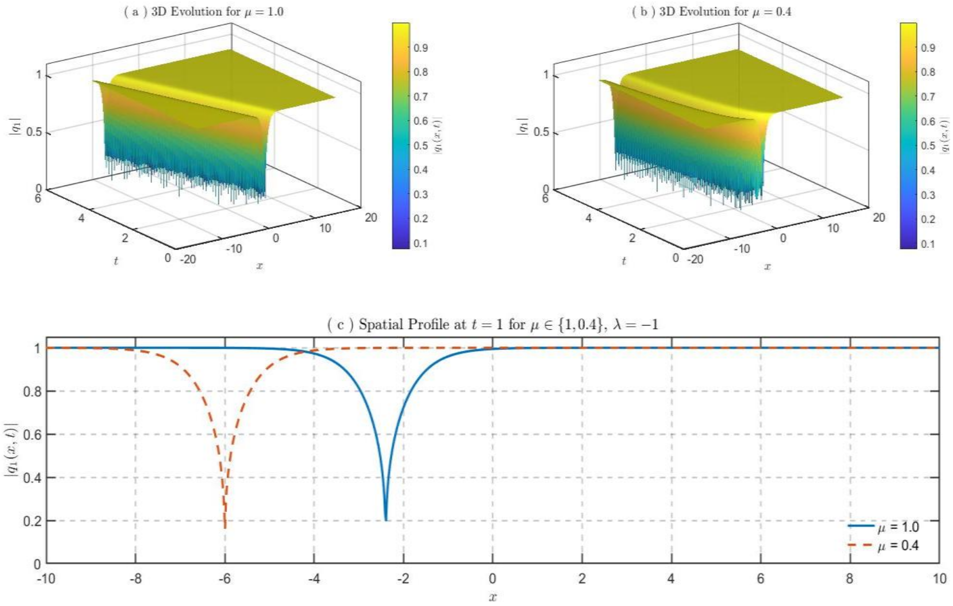

Figure 1: Spatiotemporal evolution and spatial cross-sections of the dark soliton solution $|q_1(x,t)|$ for two different conformable fractional orders. Panels (a) and (b) present the three-dimensional evolution of the soliton amplitude for $\mu = 1.0$ (classical model) and $\mu = 0.4$ (fractional model), respectively, using the parameter set $n = 3, a = 1.2, k = 1, \lambda = -1, A_0 = 0$, and $A_1 = 1$. The traveling-wave coordinate is $\zeta = x - vt^{\mu}/\mu$, where the wave velocity is $v = -2ak = -2.4$. Panel (c) compares the spatial profiles at the fixed time $t = 1$. Both cases exhibit the characteristic dark soliton depression on a finite background. Reducing the fractional order from $\mu = 1$ to $\mu = 0.4$ shifts the soliton center through the fractional traveling-wave transformation while preserving the localized dark-soliton structure and its asymptotic amplitude. The figure demonstrates that the conformable fractional-order parameter primarily modifies the propagation dynamics rather than the intrinsic waveform, illustrating the influence of fractional temporal evolution on dark-soliton propagation.

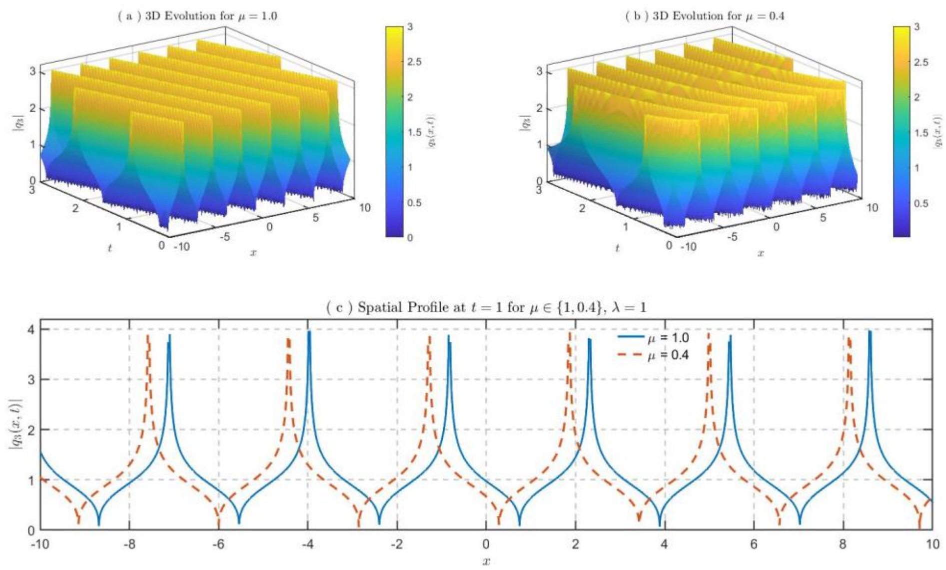

Figure 2: Spatio-temporal evolution and spatial cross-sections of the periodic wave solution

$|q_3(x,t)|$ obtained from the Riccati equation for $\lambda > 0$. Panels (a) and (b) display the three-dimensional evolution of the periodic solution for the classical case ($\mu = 1$) and the conformable fractional case ($\mu = 0.4$), respectively, using the parameter set $n = 3, a = 1.2, k = 1$, $\lambda = 1, A_0 = 0$, and $A_1 = 1$. The corresponding traveling-wave velocity is $v = -2ak = -2.4$, and the wave propagates according to the conformable fractional traveling-wave coordinate $\zeta = x - vt^{\mu}/\mu$. Panel (c) compares the spatial profiles at the fixed time $t = 1$. The solution exhibits the characteristic periodic structure generated by the tangent function, with regularly spaced singular peaks corresponding to the vertical asymptotes of $\tan(\zeta)$. For visualization purposes, the amplitudes near the singularities are truncated in the three-dimensional surfaces and omitted in the two-dimensional profiles to avoid artificial connections across the asymptotes. The comparison demonstrates that reducing the fractional order modifies the propagation dynamics and phase shift of the periodic pattern while preserving its periodicity and singular wave structure.

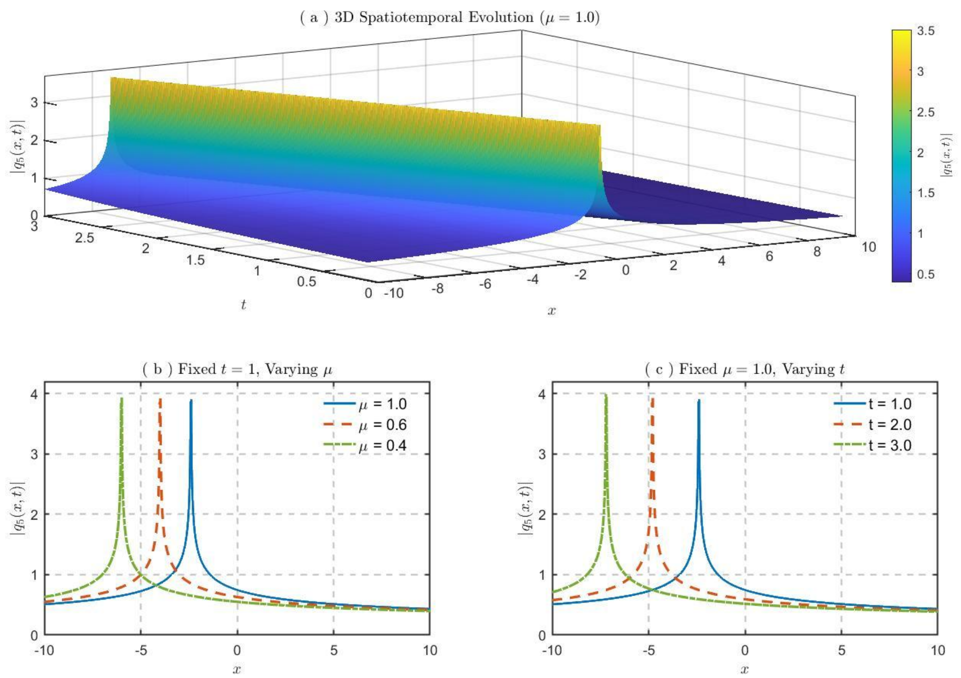

Figure 3: Comprehensive visualization of the exact rational wave solution $|q_5(x,t)|$ corresponding to the degenerate Riccati case $\lambda = 0$. Panel (a) presents the three-dimensional spatiotemporal evolution of the solution for the classical conformable order $\mu = 1$, computed using the parameter set $n = 3, a = 1.2, k = 1, A_0 = 0$, and $A_1 = 1$, which yields the traveling-wave velocity $v = -2ak = -2.4$. The solution propagates according to the conformable traveling-wave coordinate $\zeta = x - vt^{\mu}/\mu$ and exhibits a localized algebraic singularity along the trajectory $\zeta = 0$. Panel (b) compares the spatial profiles at the fixed time $t = 1$ for three fractional orders, $\mu = 1.0$, 0.6, and 0.4, illustrating that decreasing the fractional order produces a translation of the singular peak through the modified traveling-wave coordinate while preserving the characteristic algebraic decay away from the singularity. Panel (c) displays the spatial profiles for the classical case $\mu = 1$ at three different propagation times, $t = 1,2$, and 3, demonstrating that the singularity propagates with constant velocity without changing its intrinsic rational profile. For graphical clarity,

 |

amplitudes exceeding the prescribed visualization threshold are truncated in the three-dimensional surface and omitted near the singularity in the two-dimensional profiles to avoid artificial line segments across the vertical asymptote. These results confirm that the conformable fractional-order parameter primarily governs the propagation dynamics of the rational wave, whereas the temporal evolution translates the singular structure without altering its fundamental algebraic nature.

7.3 Comparison with previous work

For the corrected coefficients, the dark family q_1 in Equation (40) agrees with the family reported in [14]; the singular family q_2 matches the corresponding result there as well. The trigonometric families q_3, q_4 correspond to the periodic solutions reported in [9] for the cubic-quartic Kudryashov model and reduce to it in the appropriate parameter limit. The rational family q_5 does not appear in the comparison literature and is reported here, to our knowledge, for the first time for this exact model. We caution, however, that under any of the Riccati templates in Equation (28) the resulting families are connected by the natural limits $\lambda \to 0$ and $\lambda \to \pm\infty$, so the notion of "novel" should not be overstated.

7.4 Limitations

We have not carried out a linear-stability analysis of the obtained profiles, nor have we addressed the integrability of Equation (5) for any specific parameter sub-region; both would be natural follow-up studies.

8 Conclusion

We have presented a complete and self-checked derivation of travelling-wave solutions of the time fractional NLSE Equation (5) with Kudryashov polynomial SPM and dual nonlocal nonlinearity. Two algebraic errors in earlier treatments-a factor of two in the wavenumber-dispersion coefficient and a spurious factor of a in the nonlocal contributions to the reduced ODE, were identified and corrected, with the algebraic system re-solved accordingly. Applying the generalised tanh-coth expansion method with the Riccati auxiliary $Y' = \lambda + Y^2$ and balance number $M = 1$ produced dark, singular, periodic, and rational solution families together with explicit existence constraints. The fractional order μ controls the temporal scaling of the wave variable and serves as a tuning knob for the speed of pulse evolution, consistent with the "traffic-light" interpretation in fibre-optic transmission. Several extensions are natural: variable-order conformable derivatives, perturbation analyses, polarisation-preserving fibre and coupler geometries, and PINN-based numerical verification.

A Plot-generation script

A self-contained Mathematica script regenerating Figure 1 (c):

```
(* Mathematica *)
(* Clear any previous definitions *)

ClearAll[zeta, theta, q1, A0, A1, omega, v, mu]

(* Parameters *)
```



```
n = 3;   a = 1.2;   c1 = 1.0;   c5 = 0.1;   c6 = 0.01;   u0 = 0.1;
k = 1.;  lam = -1.;   A1 = 1.;   A0 = 0.;
(* Derived quantities *)
omega = -a k^2 - 2 a lam / n^2;
v = -2 a k;
(* Definitions *)
zeta[x_, t_, mu_] := x - v * t^mu / mu;
theta[x_, t_, mu_] := -k * x + omega * t^mu / mu + u0;
q1[x_, t_, mu_] := Surd[A0 - A1 * Sqrt[-lam] * Tanh[Sqrt[-lam] * zeta[x, t, mu]], n] * Exp[I
* theta[x, t, mu]];
(* Plot *)
Plot[{Abs[q1[x, 1, 1.]], Abs[q1[x, 1, 0.4]]}, {x, -10, 10},
 PlotLegends -> {"mu=1", "mu=0.4"},
 AxesLabel -> {"x", "|q_1|"},
 PlotRange -> All,
 ImageSize -> Medium]
```

B Case I residual constraints (long forms)

Substituting Equations (32), (34), and (36) into (38) into the Y^1 and Y^0 The coefficients of the algebraic system yield identities (after expansion and simplification), confirming that no further independent constraint arises beyond those listed in Section 6. The explicit symbolic check is contained in the script of Section A.

Appendix: Case II treated for completeness

With $V = A_0 + B_1 Y^{-1}, V' = -B_1 Y^{-2}(\lambda + Y^2) = -B_1(\lambda Y^{-2} + 1)$, and the higher derivative V'' computed analogously, the algebraic-system reduction parallels Case I. The Y^{-6} coefficient yields

$$c_4 = -\frac{6c_6\lambda^2}{B_1^2}$$

and the remaining constraints take the form of Equations (34) and (36) to (38) with the replacements $(A_0, A_1) \to (A_0, B_1/\lambda)$ (when $\lambda \neq 0$). The case $\lambda = 0$ is recovered from Equation (44) directly.

References

[1] Altwaty, A. A. (2023). Optical solitons in fiber Bragg gratings for the coupled form of the nonlinear (2+1)-dimensional Kundu–Mukherjee–Naskar equation via four powerful techniques. *Results in Physics, 44*, Article 106205.

[2] Altwaty, A. A., et al. (2020). Soliton and wave solutions to the extended Gerdjikov–Ivanov equation in DWDM system with auxiliary equation method. *Mathematical Sciences Letters, 9*(3), 57–63.

 |

[3] Altwaty, A. A., et al. (2021). Optical solitons with rational temporal evolution in fiber Bragg gratings with generalized anti-cubic nonlinearity by the fractional Riccati method. *Results in Physics, 22*, Article 103872.

[4] Ali, K., Yusuf, A., Alquran, M., & Tarla, S. (2023). New physical structures and patterns to the optical solutions of the nonlinear Schrödinger equation with higher dimension. *Communications in Theoretical Physics, 75*, 085003.

[5] Arnous, A. H., Murad, M. A. S., Biswas, A., et al. (2023). Optical solitons with cubic–quartic dispersive reflectivity having parabolic and quadratic–cubic laws of nonlinearity. *Optik, 293*, 171550.

[6] Biswas, A., & Pati, G. S. (2011). Mathematical theory of slow-light optical solitons. *Waves in Random and Complex Media, 21*(3), 456–468.

[7] Gepreel, K. A., Zayed, E. M. E., Alngar, M. E. M., et al. (2021). Optical solitons with Kudryashov's arbitrary form of refractive index and generalised non-local nonlinearity. *Optik, 243*, 166723.

[8] Gonzalez-Gaxiola, O., Biswas, A., & Belic, M. R. (2019). Optical soliton perturbation of the Fokas–Lenells equation by the Laplace–Adomian decomposition algorithm. *Journal of the European Optical Society-Rapid Publications, 15*, 13.

[9] Khalil, R., Al Horani, M., Yousef, A., & Sababheh, M. (2014). A new definition of fractional derivative. *Journal of Computational and Applied Mathematics, 264*, 65–70.

[10] Kudryashov, N. A. (2022). Method for finding optical solitons of generalised nonlinear Schrödinger equations. *Optik, 261*, 169163.

[11] Mathanaranjan, T. (2023). Optical soliton, linear stability analysis and conservation laws via multipliers to the integrable Kuralay equation. *Optik, 290*, 171266.

[12] Mathanaranjan, T., Kumar, D., Rezazadeh, H., & Akinyemi, L. (2022). Optical solitons in metamaterials with third- and fourth-order dispersions. *Optical and Quantum Electronics, 54*(5), 271.

[13] Murad, M. A. S. (2025). Formation of optical soliton wave profiles of nonlinear conformable Schrödinger equation in weakly non-local media: Kudryashov auxiliary equation method. *Journal of Optics, 54*, 3177–3190.

[14] Murad, M. A. S. (2025). Optical solutions for the perturbed conformable Fokas–Lenells equation via Kudryashov auxiliary equation method. *Modern Physics Letters B, 39*(7), 2450418.

[15] Murad, M. A. S., Arnous, A. H., Biswas, A., et al. (2023). Optical soliton perturbation with Kudryashov's quintuple-power law of nonlinearity having nonlinear chromatic dispersion. *Optik, 287*, 171106.

[16] Murad, M. A. S., Arnous, A. H., Biswas, A., et al. (2024). Suppressing internet bottleneck with Kudryashov's extended version of self-phase modulation and fractional temporal evolution. *Journal of Optics.*

[17] Murad, M. A. S., Faridi, W. A., Iqbal, M., et al. (2024). Analysis of Kudryashov's equation with conformable derivative via the modified Sardar sub-equation algorithm. *Results in Physics, 60*, 107678.

[18] Murad, M. A. S., Hamasalh, F. K., Arnous, A. H., et al. (2026). Optical solitons with Kudryashov's form of local and non-local self-phase modulation structure having

fractional temporal evolution. *Semiconductor Physics, Quantum Electronics & Optoelectronics, 29*(1), 80–90.

[19] Saleh, M. H., & Altwaty, A. A. (2020). Optical solitons of the extended Gerdjikov–Ivanov equation in DWDM system by extended simplest equation method. *Applied Mathematics & Information Sciences, 14*(5), 1–7.

[20] Saleh, M. H., & Altwaty, A. A. (2022). Solitons and other solutions to the extended Gerdjikov–Ivanov equation in DWDM system by the exp(-ϕ(ζ))-expansion method. *Ricerche di Matematica*, 1–14.

[21] Wazwaz, A.-M. (2008). The extended tanh method for new compact and non-compact solutions for the KP-BBM and the ZK-BBM equations. *Chaos, Solitons & Fractals, 38*(5), 1505–1516.

[22] Wazwaz, A.-M. (2009). *Partial differential equations and solitary waves theory*. Springer.

[23] Yildirim, Y., Biswas, A., Triki, H., et al. (2021). Cubic–quartic optical soliton perturbation with Kudryashov's law of refractive index having quadrupled-power law and dual form of generalised nonlocal nonlinearity by sine-Gordon equation approach. *Journal of Optics, 50*, 593–599.

[24] Zayed, E. M. E., Shohib, R. M. A., Alngar, M. E. M., et al. (2021). Optical solitons and conservation laws associated with Kudryashov's sextic-power-law nonlinearity of refractive index. *Ukrainian Journal of Physical Optics, 22*(1), 38–49.

