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The Fundamental Theorem of Algebra

Amani Khalid Saleh Atia *
Department of Mathematics, Faculty of Science, University of Zawia, Zawia, Libya
*Corresponding: a.atie@zu.edu.ly

المبرهنة الأساسية في الجبر

أماني خالد صالح عطية *
قسم الرياضيات، كلية العلوم، جامعة الزاوية، ليبيا

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Abstract:
The Fundamental Theorem of Algebra (FTA) asserts that every non-constant single-variable polynomial with complex coefficients possesses at least one complex root. This theorem serves as a cornerstone in mathematical discourse, bridging classical algebra and complex analysis. Despite its importance, the theorem admits various proofs rooted in different mathematical disciplines. This paper aims to provide a comprehensive exploration of the FTA by examining two distinct analytical approaches. First, we present a proof derived from advanced calculus, utilizing the extreme value theorem and properties of continuous functions on compact sets. Second, we examine a proof based on complex analysis, specifically leveraging Liouville's Theorem. By delineating these methodologies, this study highlights the profound interplay between polynomial theory and higher-level mathematical analysis, offering a structured pedagogical framework for understanding the existence of roots in the complex plane.

Keywords: Fundamental Theorem of Algebra; Complex Polynomials; Liouville's Theorem; Advanced Calculus; Complex Analysis.

الملخص
تنص المبرهنة الأساسية في الجبر على أن كل متعددة حدود غير ثابتة ذات متغير واحد ومعاملات مركبة تمتلك جذراً مركباً واحداً على الأقل. تُعد هذه المبرهنة حجر الزاوية في الخطاب الرياضي، حيث تربط بين الجبر الكلاسيكي والتحليل المركب. وعلى الرغم من أهميتها، فإن المبرهنة تقبل براهين متنوعة متجذرة في تخصصات رياضية مختلفة. تهدف هذه الورقة إلى تقديم استكشاف شامل للمبرهنة من خلال فحص نهجين تحليليين متميزين. أولاً، نقدم برهاناً مشتقاً من حساب التفاضل المتقدم، مستخدمين مبرهنة القيم القصوى وخصائص الدوال المستمرة على المجموعات المتراصة. ثانياً، نفحص برهاناً يعتمد على التحليل المركب، وتحديداً بالاستفادة من مبرهنة ليوفيل. من خلال تحديد هذه المنهجيات، يسلط هذا البحث الضوء على التفاعل العميق بين نظرية متعددات الحدود والتحليل الرياضي المتقدم، موفراً إطاراً تعليمياً منظماً لفهم وجود الجذور في المستوى المركب.

الكلمات المفتاحية: المبرهنة الأساسية في الجبر؛ متعددات الحدود المركبة؛ مبرهنة ليوفيل؛ حساب التفاضل المتقدم؛ التحليل المركب.

INTRODUCTION

The Fundamental Theorem of Algebra first appeared early in the 17th century in the writings of several mathematicians published works. Since then, however, the theorem is of such importance that it becomes known as the fundamental theorem in a branch of mathematics. The Fundamental Theorem of Algebra (FTA) encompasses one of the big ideas in mathematical discourse and has led to its own branch of mathematics, mainly the branch involving roots of polynomials.

Mathematically speaking, the Fundamental Theorem of Algebra states that every non-constant single-variable polynomial with complex coefficients has at least one complex root. Historically, finding the roots of a polynomial has been the motivation for both classical and modern algebra. In an effort to address this, considerable researchers have examined and identified the series of proofs of the Fundamental Theorem of Algebra in order to highlight how in mathematics there are many approaches to prove a single theorem.

Important key points to remember is that, one of the most famous proofs of the Fundamental Theorem of Algebra involves Liouville's theorem which states that a bounded entire function is always constant. On the other hand, the result from advanced calculus has been also classified in the literature review as a proof for the theorem. A majority of this project has been dedicated to shed light on those particular proofs of the theorem.

Project Objectives

In brief, the following are the objectives of this project:

1) To study the Fundamental Theorem of Algebra.
2) To give two different proofs for the Fundamental Theorem of Algebra.

PRELIMINARIES

In this first, literature reviews on some relevant fundamental concepts on sets and functions in which both are going to be used later in this project are explained.

Sets

Interval

A subset I of $\mathbb{R}$ having at least two elements is called an interval if whenever a,b $\in$ I and a $\leq x \leq$ b, then $x \in$ I.

Let a, b $\in \mathbb{R}$ and a < b. then

i. The set $\{x \in \mathbb{R} | \ a < x < b\}$ is called an open interval and is denoted by (a,b).

ii. The set $\{x \in \mathbb{R} | \ a \leq x \leq b\}$ is called closed interval and is denoted by [a,b].

iii. The set $\{x \in \mathbb{R} | \ a \leq x < b\}$ and $\{ x \in \mathbb{R} | \ a < x \leq b \}$ are called semi-open (or semi closed) intervals and are respectively denoted by [a,b) and (a,b].

The above four intervals are called finite intervals; each of them have length b-a. The word finite means the length of interval is finite, though each interval contains infinitely many elements.

Set Bounded Above

A subset S of $\mathbb{R}$ is said to be bounded above if there exists a real number K such that $x \leq$ K for all x in S

The real number K is called an upper bound of S.

Example

iv. The set (1,2) , [1,2] , [1,2) , (1,2] are all bounded above by 2.

v. The set of negative integers is bounded above by 0. □

Set Bounded Below

A subset S of $\mathbb{R}$ is said to be bounded below if there exists a real number K such that $x \geq$ K for all x in S.

The real number K is called a lower bound of S.

Example

The sets (1,3) , [1,3], [1,3), (1,3] and are all bounded below. □

Bounded Set

A subset S of $\mathbb{R}$ is said to be a bounded set if it is bounded below as well as above, i.e, if there exist two real numbers k and K such that $k \leq x \leq K$ for all $x \in S$. Equivalently, a subset S of $\mathbb{R}$ is bounded if there exist a real number M such that $|x| \leq M$ for all x in S.

Examples

vi. The sets (0,1) , (0,1] , [0,1] , (0,1] are bounded.

vii. Every finite set is bounded

viii. The set {½, ⅓, ¼…} is bounded. □

Unbounded Set

A subset S of $\mathbb{R}$ which is not bounded, is called an unbounded set.

Example

ix. , $\mathbb{Z}$, B and $\mathbb{R}$ are unbounded sets.

x. Set of all prime numbers is an unbounded set.

xi. $[0, \infty)$ is an unbounded set. □

Remarks

i. If a set is bounded above, then it has infinitely many upper bounds in as much as every number greater than an upper bound is also an upper bound.

ii. If a set is bounded below, then it has infinitely many lower bounds in as much as every number smaller than a lower bound is also a lower bound.

Least Upper Bound

Let S be a non-empty subset of R. A real number u is said to be a least upper bound of S if

xii. u is an upper bound of S.

xiii. For any other upper bound u' of S, u ≤ u'.

Example

i. If S = $\{\frac{1}{n} | n \in \ \}$, then a least upper bound of S is 1 and 1 ∈ S.

ii. If S = (0,1] or [0,1], then a least upper bound of S is 1 and 1 ∈ S.

iii. If S = (0,1), then a least upper bound of S is 1 and 1 Ø S. □

Remark A least upper bound of a set, if exists, is unique.

Greatest Lower Bound (Infimum)

Let S be a non-empty subset of ℝ. A real number L is called a greatest lower bound (g.l.b) of S if

xiv. L is a lower bound of S

xv. If L' is another lower bound of S, then L' ≤ L

Remarks

A greatest lower bound of a set if exists, is unique.

Example

i. In , 1 is the greatest lower bound and it belongs to

ii. If S = { $\frac{1}{n}$ | n ∈ }, then the g.l.b of S is 0 and 0 Ø S

iii. If S = [0,1) or [0,1], then the g.l.b of S is 0 and 0 ∈ S. □

Absolute Value of Real Number

The absolute value or (numerical value) of a real number x is denoted by |x| and is a real number defined by

$$|x| = \begin{cases} x \geq 0 \\ -x \ \ if \ \ x < 0 \end{cases}$$

Clearly,

i. $|x| \geq 0 \ \forall \ x$

ii. $|x| \geq x$ and $|x| \geq -x$

In fact, $|x|$ = max. $\{x,-x\}$, where max. $\{x,-x\}$ denotes the greater number between x and x.

Some Results for Absolute Value

Let x, y ∈ ℝ. Then

xvi. $|x| = 0 \Leftrightarrow x = 0$

xvii. $|x| > 0$ for $x \neq 0$

xviii. $|-x| = |x|$

xix. $|x|^2 = x^2$

xx. $|xy| = |x| . |y|$

xxi. (Triangle inequality) $|x+y| \leq |x| + |y|$

xxii. $|x-y| \geq |x| - |y|$

xxiii. $|x-y| > K, K>0 \Leftrightarrow x-k < y < x+k$

xxiv. $|x-y| \leq |x|+ |y|$

xxv. $|x| < k \Leftrightarrow -k < x < k$

Neighborhood

Let a be a point in ℝ and $\epsilon > 0$. The open interval $(a-\epsilon , a+\epsilon)$ centered at a. is called the ϵ-neighborhood of a and is denoted by J_ε (a).

Interior of a Set

Let A be a subset of ℝ. A point a is an interior point of A if there is ϵ- neighborhood of a lies entirely in A, that is, a is an interior point of A if and only if there exist $\epsilon > 0$ such that $J_\varepsilon(a) \subseteq A$. The set of all interior points of A is denoted by A° and is called the interior of A.

<u>*Example*</u>

xxvi. The interior of the set B of rational numbers is empty, because no open interior contains only rational numbers.

xxvii. The point 0 is not an interior point of the interval [0,1), because $J_\varepsilon(0)$ contains points not in [0,1). ◻

Open Set

A subset U of $\mathbb{R}$ is open if $U^{\circ} = U$. That is, a set is open if and only if every point of the set is an interior point of the set.

Example

xxviii. The empty set is an open set.

xxix. The interval (0, 2) is an open set. ◻

Closed Set

Let S be a subset of $\mathbb{R}$. Then, S is said to be closed if and only if its complement $\mathbb{R}\backslash S$ is open.

Example

xxx. $\mathbb{R}$ is closed set since $\mathbb{R}\backslash\mathbb{R} = \emptyset$ is open.

xxxi. [0, 1] is closed but not open. Because points 0,1 are not interior points of the interval [0,1].

xxxii. (0,1] is neither closed nor open. Because the point 1 is not an interior point of the interval (0,1] and complement of the interval (0,1] is not an open. ◻

Compact Set

A subset S of $\mathbb{R}$ is compact if and only if it is closed and bounded.

Functions

A function $f : X \to Y$ between sets X, Y assigns to each $x \in X$ a unique element $f(x) \in Y$. Functions are also called maps, mappings, or transformations. The set X on which f is defined is called the domain of f and the set Y in which it takes its values is called the codomain. We write $f : x \to f(x)$ to indicate that f is the function that maps x to $f(x)$.

Example

The identity function id_X: X → X on a set X is the function id_X: $x \to x$ that maps every element to itself. □

Example

Let A ⊂ X. The characteristic (or indicator) function of A, χ_A: X → {0, 1}, is defined by

$$\chi_A(x) = \begin{cases} 1 \ if \ x \ \subset A \\ 0 \ if \ x \ \emptyset \ A \end{cases}$$

□

Example

The square function f : → is defined by

$$f(n) = n^2,$$

which also can be written f : n → n^2. The equation g(n) = $\sqrt{n}$, where $\sqrt{n}$ is the positive square root function. □

Limits

Definition: Let f : A → ℝ, where A ⊂ ℝ, and suppose that c ϵ ℝ. Then

$$\lim_{x \to c} f(x) = L$$

if for every ϵ > 0 there exists a δ > 0 such that 0 < |x − c| < δ implies that |f(x) − L| < ϵ.

Example

If f(x) = x + 3, prove that,

$$\lim_{x \to 1} f(x) = 4.$$

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} (x + 3).$$

For any ϵ > 0, let δ = ϵ. Then for x satisfying

$$|f(x) - 4| = |(x + 3) - 4|$$

$$= |x - 1| < \delta = \epsilon.$$ □

Continuity

Definition

Let f : A→ ℝ, where A ⊂ ℝ, and suppose that c ∈ A. Then f is continuous at c if for every ϵ > 0 there exists a δ > 0 such that |x – c| < δ and x ϵ A implies that |f(x) – f(c)| < ϵ.

A function f : A→ ℝ is continuous on a set B ⊂ A if it is continuous at every point in B, and is continuous if it is continuous at every point of its domain A.

Example

If f : (a,b) → ℝ is defined on an open interval, then f is continuous on (a,b) if and only if

$$\lim_{x\to c} f(x) = f(c) \; for \; every \;\; a \; < c < b$$

Example

If f : [a,b] → ℝ is defined on a closed, bounded interval, then f is continuous on [a,b] if and only if

$$\lim_{x\to c} f(x) = f(c) \qquad\qquad for \; every \;\; a \; < c < b$$

$$\lim_{x\to a^*} f(x) = f(a) \qquad\qquad \lim_{x\to \bar{b}} f(x) = f(b).$$

□

Example

The function f : [0,∞) → ℝ defined by f(x) = $\sqrt{x}$ is continuous on [0,∞). To prove that f is continuous at c > 0, we note that for 0 ≤ x < ∞,

$$|f(x) - f(c)| = |\sqrt{x} - \sqrt{c}| = \left|\frac{x-c}{\sqrt{x}+\sqrt{c}}\right| \leq \frac{1}{\sqrt{c}}\,|x - c|,$$

So given $\epsilon > 0$, we can choose $\delta = \sqrt{c\epsilon} > 0$ in the definition of continuity. To prove that f is continuous at 0, we note that if $0 \leq x < \delta$ where $\delta = \epsilon^2 > 0$, then

$$|f(x) - f(0)| = \sqrt{x} < \epsilon. \qquad \square$$

COMPLEX NUMBER

FIELDS

The field F is a set with two binary operations, addition, denoted by + and multiplication denoted by or just by juxtaposition, defined on it satisfying the following nine axioms:

1) Addition is commutative: a + b = b + a for each pair a, b in F.
2) Addition is associative: a + (b + c) = (a + b) + c for a, b, c ∈ F.
3) There exists an additive identity, denoted by 0, such that a + 0 = a for each a ∈ F.
4) For each a ∈ F there exists an additive inverse, denoted by -a, such that a + (-a) = 0.
5) Multiplication is associative: a(bc) = (ab)c for a, b, c ∈ F.
6) Multiplication is distributive over addition: a(b + c) = ac + ac for a, b, c ∈ F.
7) Multiplication is commutative: ab = ba for each pair a, b in F.
8) There exists multiplicative identity, denoted by 1, (not equal to 0) such that a1 = a for each a in F.
9) For each a ∈ F, with a ≠ 0 there exists a multiplicative inverse, denoted by a^{-1}, such that $aa^{-1} = 1$.

A set G with the operation + on it and satisfying axioms (1) through (4) is called an abelian group. Axioms (1) through (6) define a ring, while Axioms (1) through (8) define a commutative ring with an identity. Therefore, in a more general algebraic context, field can be defined as a commutative ring with an identity in which every nonzero element has a multiplicative inverse.

A field can be considered as the most basic algebraic structure in which all the arithmetic operations: addition, subtraction (addition of additive inverses), multiplication, and division (multiplication by multiplicative inverses) can be done.

Examples of fields include the set of rational numbers B, the integers modulo $\mathbb{Z}_p$ for any prime P, and the field of real numbers $\mathbb{R}$.

The Complex Number Field

In the set $\mathbb{R}$ of real numbers, $\sqrt{-1}$ does not exist. Now we formally define i = $\sqrt{-1}$, that is, i is a new element such that $i^2 = -1$. Historically, i is called the imaginary unit, but as we will see in the next section, i has a very real geometric significance.

A complex number z is a number with expression of the form $z = x + i\,y$ with x, y $\in \mathbb{R}$. The x and y are respectively called the real and imaginary parts of z. If $x = 0$, $y \neq 0$, so that z has the form iy, then z is called a (purely) imaginary number. The set of complex numbers is denoted by $\mathbb{C}$, that is,

$$\mathbb{C} = \{x + iy; x, y \subset \mathbb{R}, i^2 = -1\} \bullet$$

If we identify a real number x with the complex number $x + 0$ i, we see that $\mathbb{R} \subset \mathbb{C}$ as set.

On $\mathbb{C}$, we define arithmetic by algebraic manipulation using the fact that $i^2 = -1$. That is, if $z = x + iy$, w = a + i b then:

i. $z \pm w = (x \pm a) + i(y \pm b)$.

ii. $zw = (x+iy)(a+ib) = (ax - yb) + i(xb + ya)$

$$= ax + i(yc) + i(xb) - \mathbf{i^2}(yb) = (ax - yb) + i(xb + ya)).$$

Example

Let z = 3 + 4 i and w = 7 – 2 i Then:

i. z + w = 10 + 2 i.

ii. z – w = -4 + 6 i.

iii. zw = 29 + 22 i. □

Definition

If z ∈ ℂ with z = x + i y, then the complex conjugate of z, denoted by $\overline{z}$, is the complex number $\overline{z}$ = x - iy. The absolute value, or modulus, of z, denoted by |z|, is the real number |z| = $\sqrt{x^2 + y^2}$

Example

Let z = 3 + 4 i. Then $\overline{z}$ = 3 – 4 i and |z| = $\sqrt{9 + 16}$ = 5. □

The following lemmas give the properties of the complex conjugate and the absolute value.

Lemma

Let z,w ∈ ℂ. Then

i. |z| ≥ 0, and |z| = 0 if and only if z = 0.

ii. |zw| = |z| |w|.

iii. |z + w| ≤ |z| + |w|.

Lemma

Let z,w ∈ ℂ. *Then*

iv. $\overline{z + w} = \bar{z} + \bar{w}$.

v. $\overline{zw} = \bar{z}\,\bar{w}$

vi. $\bar{\bar{z}} = z$.

vii. $|\bar{z}| = |z|$.

viii. $\bar{z} = z$ if and only if z real.

Lemma

If z ∈ ℂ, then $z\bar{z} = |z|^2$.

Proof

$z\bar{z} = (x + iy)\,(x - iy) = x^2 - i^2 y^2 = x^2 + y^2 = |z|^2$. #

Remark

From this lemma we can see that if z ≠ 0, then

$$z\,\frac{\bar{z}}{|z|^2} = \frac{|z|^2}{|z|} = 1$$

Definition

$\frac{1}{z} = \frac{\bar{z}}{|z|^2}$ for any z ∈ ℂ with z ≠ 0.

Example

Let z = 3 + 4 i and w = 7 – 2 i. Then

i. $\frac{1}{z} = \frac{3-4\,i}{3^2+4^2} = \frac{3}{25} - \frac{4}{25}\,i$.

ii. $\frac{z}{w} = z\,\frac{1}{w} = (3 + 4i)\,\frac{7+2i}{53} = \frac{13+34i}{53} = \frac{13}{53} + \frac{34}{53}\,i$. □

Theorem (DeMoivre's Theorem for Powers)

If $z = re^{i\theta} \in \mathbb{C}$ and $n \in$, then $z^n = r^n e^{in\theta}$. Notice then that $|z^n| = |z|^n$ and Arg $(z^n) = n$ Arg z. This is known as DeMoivre's theorem for powers.

Theorem (DeMoivre's Theorem for Roots)

If $z \in \mathbb{C}, z \neq 0$, then there are exactly n distinct nth roots of z. If $z = re^{i\theta}$, these are given by:

$$W_K = r^{1/n} e^{i\frac{(\theta+2k\ \pi)}{n}}, k = c, 1, \dots, n-1.$$

Polynomials

Since the Fundamental Theorem of Algebra deals with polynomials, it makes sense that we have some understanding of polynomial functions and their properties. Polynomials are used in a wide range of problems, from elementary word problems to calculus, chemistry, economics, physics, and numerical analysis. In order to understand the polynomials, one first needs to understand what classifies as a monomial.

Definition (Monomial)

A single term that consists of a number, a variable, or the product of a number and variable to a positive integer exponent is called a monomial.

Examples

iii. 5, -24, x, $4x$, and $3x^3$ are all monomial

iv. $x + y$, $12\,xy + z^a$ are not monomial. □

Understanding monomials helps one to understand polynomials because a monomial is a polynomial with only one term.

Definition (Polynomial)

A polynomial is the addition or subtraction of monomials written in the form:

$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots. + a_1 x^2 + a_0 x + a^n$,

where n is a non-negative integer and a^n, a_0, $a_1 \ldots a_n$ are real constants. The degree of a polynomial is the largest exponent or sum of exponents in any of its monomials.

Examples

v. $g(x) = 3x^a + 2x^3 - 6x^2 + 7x - 3$ and $h(x) = x^a + 2x - 1$ are polynomials of the degree 4.

vi. $f(x) = 4x + 5$ is a polynomial of the degree 1.

vii. $g(x) = 5x^2 + 2$, is a polynomial of the degree 2. □

Some polynomials have special names based on their degrees and they are:

Degree	***Name***	***Example***
0	Constant	$y = 7$
1	Linear	$y = 2x + 1$
2	Quadratic	$y = x^2 - 3x - 2$
3	Cubic	$y = x^3 + 4x^2 - 6$

Adding, Subtracting and Multiplying Polynomial

To add or subtract polynomials, we add or subtract the coefficients of like terms.

Example

viii. $(x^2 + 2x - 5) + (3x^3 - 4x + 8) = 3x^3 + x^2 - 2x + 3$

ix. $(x^2 + 3x) - (3x^2 + x - 4) = 2x^2 + 3x - 3x^2 - x + 4$

$$= -x^2 + 2x + 4$$

x. $(2x^2 + 5) \bullet (5x + 3) \quad = 2x^2 (5x + 3) + 5 (5x + 3)$

$$= 10x^3 + 6x^2 + 25x + 15$$ ❑

<u>Remark</u>

1) When adding or subtracting two polynomials, the degree of the new polynomial will be the same or smaller than the largest degree polynomials term in the sum.
2) When multiplying two polynomials, the degree of the new polynomial is the sum of the degree of the two polynomials.

Real and Complex polynomials

This section discusses polynomials with coefficient in the fields $\mathbb{R}$ and $\mathbb{C}$. These polynomials are respectively called the real and complex polynomials. We donated by $\mathbb{R}[x]$ and $\mathbb{C}[x]$ respectively, the sets of real and complex polynomials. We will discuss some properties which are needed later.

Theorem

A real polynomial of odd degree has a real root.

<u>Proof</u>

Suppose $p(x) \in \mathbb{R}[x]$ with deg $p(x) = n = 2k + 1$ and suppose that the leading coefficient $a_n > 0$ (the proof is almost identical if $a_n < 0$). Then $P(x) = a_n x^n$ + (lower terms). Then,

xi. $\lim_{x\to\infty} p(x) = \lim_{x\to\infty} a_n x n = \infty$ since $a_n > 0$

xii. $\lim_{x\to-\infty} (x) = \lim_{x\to-\infty} a_n x n = -\infty$ since $a_n > 0$ and n is odd.

From (i) $p(x)$, gets arbitrarily large positively so there exists an $x4$ with $p(x4) > 0$. Similarly, from (ii) there exists an x, with p(x,) < 0.

A real polynomial is a continuous real-valued function for all $x \in \mathbb{R}$. Since $P(x4)\ P(x,) <$ 0, it follows from the intermediate value theorem that there exists an x_2, between $x4$ and x,, such that $P(x_2) = 0$. #

Lemma

Every degree 2 complex polynomial has a root in $\mathbb{C}$.

Proof

If $P(x) = ax^2 + bx + c \in \mathbb{C}[x]$, then the roots formally are given by the quadratic formula:

$$x4 = \frac{-b+\sqrt{b^2-4ac}}{2a},\ x, = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

From DeMoivre's theorem (3.3) every complex number has a square root, hence $x4$, x, exist in $\mathbb{C}$. They of course, may be the same, if $b^2 - 4ac = 0$. #

To go further, there is a need to introduce the concept of the conjugate of a polynomial and some straightforward consequences of this idea.

Definition

If $p(x) = ao + \cdots + a_n x^n$ is a complex polynomial, then its conjugate is the polynomial $\bar{p}(x) = \bar{a}o + \cdots + \bar{a}_n x^n$ That is, the conjugate is the polynomial whose coefficients are the conjugates of those of $p(x)$.

Example

If $p(x) = 2x^2 + (5 - i)\,x + 3i$, then

$\bar{p}(x) = 2x^2 + (5 + i)\,x - 3i$ ❑

Lemma

For any $P(x) \in \mathbb{C}[x]$

1) $\overline{p(z)} = \bar{p}(\bar{z})$ if $z \in \mathbb{C}$
2) $p(x)$ is a real polynomial if and only if $p(x) = \bar{p}(x)$
3) If $p(x)\, Q(x) = H(x)$ then $\overline{H}(x) = (\bar{p}\ (x))(Q\,\overline{(x)})$

<u>*Proof*</u>

i. Suppose $z \in \mathbb{C}$ and $p(z) = ao + a_1 z + \ldots + a_n z^n$. Then

$$\overline{p(z)} = \overline{ao + \cdots + a_n z^n} = \overline{ao} + a4\overline{z} + \cdots + \bar{a}_n z^n = \bar{p}\,\bar{(}z).$$

ii. Suppose p(x) is real then $a_i = \bar{a}_i$ for all its coefficients and hence $p(x) = \bar{p}\,(x)$.

conversely suppose $p(x) = \bar{p}(x)$ then $a_i = \bar{a}_i$ for all its coefficients and hence $a_i \in \mathbb{R}$ for each a_i and so p(x) is real polynomial. #

Lemma

Suppose $G(x) \in \mathbb{C}[x]$. Then $H(x) = G(x)\overline{G}(x) \in \mathbb{R}[x]$.

<u>*Proof*</u>

$\overline{H}(x) = \overline{G(x)\overline{G}(x)} = \overline{G}(x)\overline{\overline{G}}(x) = G(x)\overline{G}(x) = G(x)\overline{G}(x) = H(x)$. Therefore, $H(x)$ is a real polynomial. #

Lemma

If $f(x) \in \mathbb{R}[x]$ and $f(zo) = 0$ then $f(\bar{z}o) = 0$; then complex roots of real polynomials come in conjugate pairs.

Proof

$f(z_o) = 0$ implies that $f(\overline{z_o}) = 0$ but then $\bar{f}(z_o) = 0$. Since $\bar{\bar{f}}(x)$ is real, $f(x) = \bar{\bar{f}}(x)$, so $f(\bar{z_o}) = 0$. #

Theorem

If every non-constant real polynomial has at least one complex root, then every non-constant complex polynomial has at least one complex root.

Proof

Let $p(x) \in \mathbb{C}[x]$ and suppose that every non-constant complex polynomial has at least one complex root. Let $H(x) = p(x)\bar{p}(x)$. as it was mentioned previously in lemma (3.5.5), $H(x) \in \mathbb{R}[x]$. by supposition there exists a $z_o \in \mathbb{C}$ with $H(z_o) = 0$. Then $p(z_o)\bar{p}(z_o) = 0$, and since $\mathbb{C}$ has no zero divisors, either $\bar{p}(z_o) = 0$ or $p(z_o) = 0$. In the first case z_o is a root of $\bar{p}(x)$. In the second case $p(z_o) = 0$. Therefore, $\bar{z}_o$ is a root of $p(x)$. #

3.2 Complex Functions

The function $f : \mathbb{C} \to \mathbb{C}$ is called a complex-valued function with complex variable. If $z = x + iy$ and $w = f(z)$, then we can write $w = f(z) = f(x + iy) = u(x,y) + iv(x,y)$. Then function u and v are called the real and imaginary parts of f.

Example

Consider the complex function $f(z) = z^2$. Determine its real and imaginary parts.

Suppose $z = x + iy$. Then $z^2 = (x + iy)^2 = (x^2 - y^2) + i(2xy)$. Hence here $\mathit{Ref}(z) = x^2 - y^2$ while $\mathit{Imf}(z) = 2xy$. ❑

A region is any subset of $\mathbb{C}$. A region $U \subset \mathbb{C}$ is open if for each point $zo \in U$ there exists some ϵ - neighborhood of zo entirely contained in U. A region $C \subset \mathbb{C}$ is closed if its complement C' is open.

The definition of limits for complex functions is essentially the same as for the one-variable real functions of elementary calculus.

Definition

The limit of f(z), as z approaches zo, is wo if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(z - wo)| < \epsilon$ whenever $0 < |z- zo| < \delta$. We write $\lim_{z\to -zo} (z) = wo$.

All the basic limit theorems from elementary calculus products, sums, constants, etc. carry over to this situation. To actually compute limits we pass to the real and imaginary parts of the functions.

3.6.1 Lemma

Suppose f(z) = u(z) + iv(z). Then

$$\lim_{z\to zo} f(z) = \lim_{z\to zo} u(z) + i \lim_{z\to zo} v(z).$$

Example

Let $f(z) = (x^2 + y^2) + i(2xy)$. Find $\lim_{z\to 1+i} f(z)$.

Then,

$$\lim_{z\to 1+i} f(z) = \lim_{(y,x)\to(1,1)} (x^2 + y^2) + i \lim_{(y,x)\to(1,1)} (2xy) = 2+2i. \quad ❑$$

Definition

A function $f : U \to \mathbb{C}$ is continuous at $z_0 \in U \subset \mathbb{C}$ if $\lim_{z \to z_0} f(z) = f(z_0)$. We say

f(z) is continuous on the region *U* if it is continuous at all points of *U.*

All the basic results on continuity for one-variable real functions (sums, products, etc.) carry over to complex functions. Further, as with limits.

Lemma

A function $f(z) = u(z) + iv(z)$ is continuous at $z_0 = (x_0, y_0)$ if and only $u(x,y)$ and $v(x,y)$ are continuous real-valued functions at (x_0, y_0).

Since complex polynomials are built up from only algebraic operations, considered as functions $\mathbb{C} \to \mathbb{C}$, they will be continuous everywhere in $\mathbb{C}$. Since $|z^n| \to \infty$ as $|z| \to \infty$ that is, $|f(z)| \to \infty$ as $|z| \to \infty$ for any noconstant $f(z) \in \mathbb{C}[z]$. Further, since $|f(z)|$ is then a continuous real-valued function, it is bounded on any compact region, Based on this fact, a complex polynomial as a polynomial function on $\mathbb{C}$.

Lemma

Let $f(z) \in \mathbb{C}[z]$ Then:

i. f(z) is continuous everywhere in $\mathbb{C}$

ii. $\lim_{z \to \infty} |f(z)| = \infty$ if f(z) is non-constant.

iii. $|f(z)|$ is bounded on every compact region in $\mathbb{C}$.

Derivatives and Differentiable

Definition: If f(z) is any complex function, then its derivative f'(zo) at zo ∈ ℂ is given by

$$f'(zo) = \lim_{\Delta z \to 0} \frac{(z0+ \Delta z)-(zo)}{\Delta z},$$

Whenever this limit exists, If f'(zo) exists, then f(z) is said to be differentiable at zo if said to be differentiable on U if it is differentiable at each point of U.

As with limits and continuity, all the basic differentiation rules from elementary calculus - sums, products, quotients, chain rule etc. - can be shown to be valid for complex functions.

In particular the power rule – $f(z) = z^n$ implies $f'(z) = nz^{n-1}$ for n a natural number - follows purely formally from the definition. Therefore, a complex polynomial must have a derivative at each point of ℂ.

Lemma

If $f(z) = ao + a4z + \dots + a_n z^n \in \mathbb{C}[z]$ then f'(z) exists at each point zo ∈ ℂ and $f'(zo) = a4 + \dots + na_n z_o^{n-1}$. Formally, if f(z) ∈ ℂ[z] and deg f(z) ≥ 1, then f'(z) ∈ ℂ[z] and deg f'(z) = deg f(z) – 1. If f(z) = ao is a constant, then f'(z) = 0.

Definition

Let U ⊂ ℂ be a domain. A function f : U → ℂ is analytic at zo if f is differentiable in a neighborhood of zo. We say f is analytic in U if it is analytic at each point of U. If f is analytic throughout ℂ, then it is called an entire function.

Lemma

Suppose w = f(z) is a real valued complex function. Then f(zo) exists,

$$f'(zo) = \frac{Gf}{Gx}(zo) = \frac{Gf}{Gy}(zo).$$

Proof

From the definition (3.6.3)

$$\lim_{\Delta z \to 0} \frac{(zo + \Delta z) - (zo)}{\Delta z}$$

Since f(z) is real-valued, it follows that f(z) = u(z). Then

$$f'(zo) = \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{(xo + \Delta x, o + \Delta y) - (xo, yo)}{\Delta z}$$

Since f'(zo) exists, the limit is independent of the mode of approach. Approaching along a line parallel to the real axis Δz = Δx,Δy = 0

$$f'(zo) \lim_{\Delta x \to 0} \frac{(x + \Delta x, y) - (x, y)}{\Delta x} = \frac{Gu}{Gx} = \frac{Gf}{Gx}$$

Similarly approaching along a line parallel to the imaginary axis gives the second part. #

Example

Let f(z) = $|z|^2$. Then f'(0) exists but f(z) is not analytic at z = 0.

Suppose zo = 0 and f(z) = $|z|^2$ and consider

$$\lim_{\Delta z \to 0} \frac{(zo+\Delta z)-(zo)}{\Delta z} = \lim_{\Delta z \to 0} \frac{|\Delta z|^2}{\Delta z} = 0.$$

Therefore, f'(0) exists and f'(0) = 0 but somehow, this cannot be analytic at 0.

Now, if $z = x + iy$, $|z|^2 = x^2 + y^2$ If f'(zo) exists, then f'(zo) $\frac{Gf}{Gx}$= (zo) = 2*x*o = $\frac{Gf}{Gy}$(zo) = 2yo. This then is possible only if *x*o = yo, and thus the derivative can only possibly exist along the line y = *x* and hence does not exist in a circular neighborhood of 0. Therefore, it is not analytic at 0. ❑

Theorem

If f'(z) = 0 for all z in a domain U ⊂ ℂ, then f is a constant in U.

Complex Integration

Definition

A curve ɣ in ℂ is a continuous function ɣ : [a,b] → ℂ given by

$$ɣ(t) = x(t) + iy(t),$$

where x(t) and y(t) are real valued functions on the interval [a,b].

Definition

A simple closed curve is a curve ɣ(t) : [a,b] → ℂ such that ɣ(a) = ɣ(b) but ɣ(t4) ≠ ɣ(t,) for no other pair t4, t, ∈ [a,b].

Definition

A simply connected domain U is a domain such that every simple closed curve within it encloses only points of U.

Definition

(a) Suppose f(t) = u(t) + iv(t) is a continuous complex function defined on the interval to $\leq t \leq t_1$. Then we define

$$\int_{to}^{t4} f(t)dt = \int_{to}^{t4} u(t)dt + i\int_{to}^{t4} v(t)dt.$$

(b) Suppose f(z) = u(z) + iv(z) is a continuous complex function and ɣ(t) = x(t) + iy(t), to $\leq t \leq t4$, is a curve in the domain of f(z). Then f(ɣ(t)) is a continuous complex function of the real variable t, and we define the complex contour integral, or complex line integral, by

$$\int f(z)dz = \int_{to}^{t4} f(\ (t))\ ´(t)dt.$$

Example

Evaluate $\int\ z^3\ dz,$ where ɣ is the straight line segment from (0,0) to (1,1).

Now, f(z) = z^3, with the straight line segment from (0,0) to (1,1) is given by ɣ(t) = t + it f(ɣ(t)) = (t + it)^3. Note that ɣ´(t) = 1+i

From definition 3.7.4

$$\int\ (z) = \int_{to}^{t4} f(\ (\mathrm{t}))\quad ´(\mathrm{t})\ dt$$

$$= \int_0^1 (t+it)^3(1+i)\quad dt$$

$$= \int_0^1 (t^3 - it^3 + 3it^3 - 3t^3)\ \ (1+i)\ dt$$

$$= \int_0^1 (t^3 - it^3 + 3it^3 - 3t^3 + it^3 + t^3 - 3t^3 - 3it^3)\ \ dt$$

$$= \int_0^1 (2t^3 - 6t^3)\ dt$$

$$= \int_0^1 -4t^3 dt = -1.$$

❑

Theorem (Cauchy's Theorem)

Let f(z) be analytic throughout a simply connected domain U and suppose ɣ is a closed curve entirely contained in U. Then

$$\int (zo)dz = 0.$$

Theorem (Cauchy Integral Formula)

Let f(z) be analytic in a simply connected domain U containing a simple closed curve ɣ. If zo is any point interior to ɣ, then

$$(z_0) = \frac{1}{2\pi i}\int \frac{(z)}{z - zo} dz,$$

where the integral is taken in counterclockwise direction around ɣ.

Example

find $\int \frac{dz}{z^2-9}$, where ɣ is any simple closed contour not containing z = -3 in its interior.

Let f(z) = $\frac{1}{z+3}$. If ɣ does not contain z = -3 in its interior, then f(z) is analytic on ɣ

and its interior. Therefore, from the Cauchy integral formula

$$\int \frac{dz}{z^2 - 9} = \int \frac{dz}{(z+3)(z-3)} = \int \frac{f(z)}{z-3} dz = 2(3).$$

Therefore, this value is $2\pi i/6 = \frac{\pi i}{3}$. ❑

Corollary

Let f(z) be analytic in a simply connected domain U containing a simple closed curve ɣ. If zo is interior to ɣ then an expression for the n – th derivative is

$$f(n)(zo) = \frac{n!}{2\pi i} \int \frac{f(z)}{(z-zo)n^{*1}}\ dz.$$

The Cauchy integral formula can be used to prove the Cauchy estimate, which indeed will be instrumental in conducting the second proof of the Fundamental Theorem of Algebra.

Theorem (Cauchy's estimate)

Suppose f(z) is analytic in a simply connected domain U $\subset \mathbb{C}$ containing the circle Co of radius ro centered at zo. If M is the maximum value of |f(z)| on Co, then

$$|f^{(n)}(z_0)| \leq \frac{n!\,M}{r\mathrm{n}}$$

In particular, if If(z)I < M within and on Co then

$$\mathrm{I}f'(z)\mathrm{I} < \frac{M}{ro}.$$

for all z interior to Co.

Proof

From corollary 3.7.1

$$|f(n)(zo)| \leq |\frac{n!}{2\pi i} \int_{co} \frac{f(z)}{(z-zo)n^{*1}}\ dz| \leq \frac{Mn!}{2\pi} |\int_{co} \frac{dz}{(z-zo)n^{*1}}|.$$

On Co,z = zo + roei', dz = iroei' dt, so

$$\mathrm{I} \int_{co} \frac{dz}{(z-zo)n^{*1}} \mathrm{I} = \mathrm{I}\int_0^{2\pi} r\bar{o}ne^{-int}\ dt\mathrm{I}=2\pi r\bar{o}n$$

Hence $|f(n)(zo)| \leq \frac{Mn!}{ron}$.

#

THE FUNDAMENTAL THEOREM OF ALGEBRA

In this chapter, two proofs will be given for The Fundamental Theorem of Algebra. Firstly, the first proof is based on solely on advanced calculus. The second proof is based on Liouville's theorem.

Let us recall what the Fundamental Theorem of Algebra is.

Theorem (The Fundamental Theorem of Algebra):

If $f(x) \in \mathbb{C}[x]$, with $f(x)$ non-constant, then $f(x)$ has at least one complex root.

First Proof Based on Advanced Calculus

This proof uses some knowledge from advanced calculus. The following lemme, which is the two-dimensional version of the extreme values theorem from elementary calculus, will be used.

Lemma

If $f : D \to \mathbb{R}$ is continuous, where D is a closed and bounded (compact) subset of $\mathbb{R}^2$, then $f(x,y)$ has a minimum and maximum value on D.

Lemma

Let $f(x) \in \mathbb{C}[x]$. Then $|f(x)|$ takes on a minimum value at some point $zo \in \mathbb{C}$.

Proof

It is straightforward that as $|x| \to \infty$, $|f(x)| \to \infty$. Since $|f(x)|$ is large for large $|x|$ it follows that the greatest lower bound m of $|f(z)|$ for $z \in \mathbb{C}$ is also the greatest lower bound in some sufficiently large disk $|z| \leq r$. Since $|f(x)|$ is a continuous real-valued function it follows from Lemma 4.1.1 that $|f(x)|$ will attain its minimum value on this disk. #

Lemma

Suppose $f(x) \in \mathbb{C}[x]$ with $f(x)$ nonconstant. If $f(xo) \neq 0$, then $|f(xo)|$ is not the minimum value of $|f(x)|$.

Proof

Let $f(x) = c_0+c_1x+c_2x^2+ \cdots\cdots + c_nx^n$ be a non-constant complex polynomial and suppose xo is a point with $f(xo) \neq 0$. Make the change of variable $x + xo$ for x. This shifts xo to the origin, so that we may assume that $f(0) \neq 0$. Next multiply $f(x)$ by $f(0)^{-1}$ so that $f(0) = 1$. We must then show that 1 is not the minimum value of $|f(x)|$.

Let k be the lowest nonzero power of x occurring in $f(x)$. Then $f(x)$ can be assumed to have the form

$$f(x) = 1 + ax^k + (\text{terms of degree} > k).$$

Now let $\propto$ be a k-th root of $-\bar{a}^1$, which exists by DeMoivre's theorem (3.3). Make the final change of variable $\propto x$ for x. Now $f(x)$ has the form

$$f(x) = 1 - x^k + x^{k*1}\ g(x) \text{ for some polynomial } g(x).$$

For small positive real x, we obtain from the triangle inequality

$$|f(x)| \leq |1 - x^k| + x^{k*1}\ |g(x)|.$$

But $x^k < 1$ for small real x, so this has the form

$$|f(x)| \leq 1 - x^k + x^{k*1}\ |g(x)| = 1 - x^k\ (1 - x|g(x)|).$$

For small real x, $x|g(x)|$ is small, so x_0 can be chosen so that $x_0\,|g(x_0)| < 1$. It follows that $x_0^k\,(1 - x_0|g(x_0)|) > 0$, so then $|f(x_0)| < 1 = |f(0)|$, completing the proof. #

Frist proof of Fundamental Theorem of Algebra:

Let $f(x)$ be a no-constant complex polynomial. From Lemma 4.1.2, $|f(x)|$ has a minimum value at some point $x_0 \in \mathbb{C}$. Then from Lemma 4.1.3, it follows that $|f(x_0)| = 0$, and hence $f(x_0) = 0$, otherwise it would not be the minimum value. Therefore, $f(x)$ has a complex root. #

The Second Proof Based on Liouville's Theorem

The second proof of the Fundamental Theorem of Algebra is based on Liouville's Theorem as it described below. It can be seen that Liouville's Theorem is related to the Cauchy integral formula.

Theorem (Liouville's Theorem)

Suppose $f(z)$ is entire and $|f(z)|$ is bounded for all values of $z \in \mathbb{C}$. Then $f(z)$ is a constant.

More generally, if $|^{(n)}(z)|$ is bounded throughout $\mathbb{C}$, then $f(z)$ is a polynomial of degree at most n+1.

<u>*Proof*</u>

Suppose $f(z)$ is entire and $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Then from the estimation of Cauchy on a circle of radius r centered at the origin,

$$|f'(z)| < \frac{M}{r}.$$

Since f(z) is entire, we can let $r \to \infty$. Therefore, $|f'(z)| = 0$ and hence $f'(z) = 0$. This is of course implies that f(z) must be a constant, completing part (1).

For part (2), if $|f^{(n)}(z)| \leq M$ for all $z \in \mathbb{C}$, then again from the estimation of Cauchy,

$$|f^{(n*1)}(z)| \leq \frac{M}{r}$$

On any circle of radius r centered at the origin. Again, by letting $r \to \infty$ we would obtain that $f^{(n*1)}(z) = 0$ and hence $f^{(n)}(z)$ is a constant. Then f(z) is a polynomial of degree at most n + 1 by antidifferentiation. #

The Second Proof of Fundamental Theorem of Algebra:

Suppose p(z) is a complex polynomial, and let $f(z) = \frac{1}{p(z)}$. If p(z) had no complex root, then f(z) would also be an entire function. If deg $p(z) \geq 1$ then p(z) is a nonconstant entire function with the property that $|p(z)| \to \infty$ as $|z| \to \infty$.

Since $|p(z)| \to \infty$ as $|z| \to \infty$, there exists $M, r > 0$ such that $|p(z)| > M$ if $|z| > r$. This implies that for $|z| > r$, $|f(z)| = \frac{1}{|p(z)|} < \frac{1}{M}$. If f(z) were entire, it would be continuous

and thus bounded on the compact set $|z| \leq r$. It follows that if f(z) were entire it would be bounded throughout $\mathbb{C}$. From Liouville's theorem, it would then follow that f(z) must be a constant. However, then p(z) would also be a constant, which is a contradiction. Therefore, p(z) must be zero for at least one value of $z \in \mathbb{C}$. #

Conclusion

The Fundamental Theorem of Algebra states that every non-constant single-variable polynomial with complex coefficients has at least one complex root. Based on this fact, the effort made in conducting this project was to provide two proofs for the Fundamental Theorem of Algebra for polynomials with complex coefficients. Accordingly the researcher identified and classified preliminaries and complex number through an extensive review. Subsequently, the first proof was given based on advanced calculus. In contrast, Liouville's theorem was easily obtained as a second proof of the Fundamental Theorem of Algebra.

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